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On ill-posedness and stability of tensor variational inequalities: application to an economic equilibrium

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Abstract
The general tensor variational inequalities, recently introduced in Barbagallo et al. (J Nonconvex Anal 19:711–729, 2018), are very useful in order to analyze economic equilibrium models. For this reason, the study of existence and regularity results for such inequalities has an important rule to the light of applications. To this aim, we start to consider some existence and uniqueness theorems for tensor variational inequalities. Then, we investigate on the approximation of solutions to tensor variational inequalities by using suitable perturbed tensor variational inequalities. We establish the convergence of solutions to the regularized tensor variational inequalities to a solution of the original tensor variational inequality making use of the set convergence in Kuratowski's sense. After that, we focus our attention on some stability results. At last, we apply the theoretical results to examine a general oligopolistic market equilibrium problem.

Keywords Tensor variational inequality · Noncooperative game · Ill-posedness · Stability

Mathematics Subject Classification 47A5 · 47J30 · 49J40 · 49K40 · 65K10 · 91A10

1 Introduction

A general tensor variational inequality has been introduced for the first time in [10] in which the inner product is done between two tensors. The tensor variational inequality problem is studied by many authors: for a recent literature on the subject see [5,49] and the reference therein. In the setting of tensor spaces, Song and Qi [42] presented a class of complementarity problems, called tensor complementarity problems, where the involved function is defined by some homogeneous polynomial of degree n, with $n \geq 2$. Moreover, they considered the properties of matrices like positive definiteness, P-matrix and copositivity to obtain

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some results for the linear complementarity problem. Such properties have been opportunely extended to symmetric tensors (see [15,18]). Furthermore, the property of copositivity has been generalized to tensors in [36]. Then, the tensor complementarity problem with $q \neq 0$ has been studied in [12]. In such a paper, the tensor variational inequality which expresses the complementarity problem for $X = \mathbb{R}^n_+$ is introduced. In [48], some existence and uniqueness theorems are obtained for a suitable tensor variational inequality.

The tensor variational inequality problem is an important tool to analyze a general oligopolistic market equilibrium problem in which the firms produce many goods and compete with a noncooperative behaviour. The first scholar who studied such a behaviour between two producers of a given commodity was Cournot [13]. He obtained that if both producers try, each one on his own, to maximize their respective income, they will produce certain definite quantities of the commodity for the market. Nash [32,33], extended Cournot's duopoly problem for a general model with n agents, each acting according to his own self-interest, the so-called noncooperative game. Each player has at his disposal a strategy which he chooses from a set of feasible strategies. The rationality postulate of noncooperative behavior can be stated as follows: each player chooses a strategy which maximizes his utility level given the decisions of the other players. Many scholars studied existence and uniqueness results for the noncooperative game under different assumptions (see for instance [17,21,39]). It is worth mentioning that Dafermos and Nagurney (see [14]) obtained the relationship between the oligopoly market model and the competitive spatial price equilibrium model studying the finite dimensional variational approach.

The aim of the paper is to analyze the ill-posedness of tensor variational inequalities and give some stability results for solutions to such inequalities. For what concerns the variational inequalities and their approximations, there is an extensive literature. More precisely, existence and approximations of solutions to variational inequalities for various classes of operators in Hilbert and Banach spaces have been considered by Browder [11], Stampacchia [43], Mosco [30,31], Alber [2], Bakushinskii [7], Doktor and Kucera [16], Liskovets [23], Alber and Rjazantseva [3], Rjazantseva [37,38], Liskovets [24,25], McLinden [29], Tossings [46], Gwinner [19], see also Liu [26], Liu and Nashed [27,34], the related references cited in [34], and the monographs by Tikhonov [44,45], Kaplan and Tichartschke [20], Bakushinskii and Goncharskii [8,9], Vasin and Ageev [47] and others. Browder [11] and Stampacchia [43] investigated on the convergence of solutions to variational inequalities when there is no perturbation of the convex set. Mosco [30,31] analyzed the convergence of what is called "the Mosco scheme" under a suitable condition on the operator. Doktor and Kucera [16] obtained convergence rates when the operator is strongly monotone. In that case the problem is well-posed. Several authors have examined regularization and iterative approximation for ill-posed variational inequalities. We remind, for instance, Alber, Notik, Liskovets and Rjazantseva (see [4,24,37]). Alber [37] and Rjazantseva [4] studied convergence under a condition stronger than the set convergence in Mosco's sense. Liskovets [24] considered the problem under an assumption of (S)-property. Bakushinskii [7] and Bakushinskii and Goncharskii [8] investigated on the convergence and covergence rate for iterative solutions. Versions of Mosco's perturbation and convergence scheme have been developed in several papers. In particular, we mention Tossings [46] who applied Mosco's scheme for solving ill-posed problems. Finally, we remind that Gwinner [19] studied variational inequalities with pseudomonotone functions on noncompact sets and established the relation between the existence of solutions under noncoercive assumptions and the convergence of an abstract regularization procedure.

In [10] the authors studied variational inequalities defined on a class of structured tensors of which an important special case is the tensor version of the nonlinear complementarity problem. Moreover they introduced the general tensor variational inequality problem and they proved only initial results concerning existence and uniqueness of solutions to such a problem. This class of variational inequalities permits to express the equilibrium condition of a more realistic oligopolistic market equilibrium problem in which every firm produces different commodities.

In this paper, we continue the study of tensor variational inequalities done in [10] proving new properties of the solution to a tensor variational inequality. In particular the new contributions concern the convergence of solutions of a family of regularized tensor variational inequalities to a solution of an ill-posed tensor variational inequality. We analyze the above problem with the Kuratowski's approximation of closed convex sets. Moreover, we study the stability of the solution to a tensor variational inequality. More precisely, we obtain a result that establishes that a small change in the tensor function produces a small change on the solution. The previous theoretical results are applied to the general oligopolistic market equilibrium problem.

The paper is organized as follows. In Sect. 2, we give some preliminary results for tensor variational inequalities. In Sect. 3, we study the convergence of solutions to an ill-posed tensor variational inequality. In Sect. 4, we establish a sensitivity result that shows how the equilibrium solution can change if the data have been perturbed. In Sect. 5, we present the general oligopolistic market equilibrium problem. In Sect. 6, a numerical example is provided.

2 Preliminars

We recall some preliminars about the Hilbert space of tensors. If we denote by V a finite dimensional vector space and by $\langle \cdot, \cdot \rangle$ his inner product, a N-order tensor is an element of the product space $V \times \cdots \times V$, i.e. a multidimensional array. Low order tensor are known as matrices (tensors of order two), vectors (tensors of order one) and scalars (tensors of order zero).

We denote with $\mathcal{T}_{N,m}(V)$ the set of all the order N tensors on the m-dimensional vector space V. A N-order tensor on a vector space V of dimension m has m^N entries. We write shortly $\mathcal{T}_{N,m}$ for tensors on the Euclidean space $\mathbb{R}^m$ and when the dimensions are clear in the context, simply $\mathcal{T}$. A tensor $\mathbf{A}$ of order N is indicated by its entries, namely the element $(i_1, i_2, \ldots, i_N)$ of $\mathbf{A}$, belongs to $\mathbb{R}$ is denoted by $a_{i_1,i_2,\ldots,i_N}$.

We define the following inner product $\langle\langle \cdot, \cdot \rangle\rangle$ on $\mathcal{T}_{N,m}$ endowing it to a structure of Hilbert space.

Definition 1 Let $\mathbf{A}, \mathbf{B}$ be two tensors in $\mathcal{T}_{N,m}$. Let us define the application $\langle\langle \cdot, \cdot \rangle\rangle : \mathcal{T}_{N,m} \times \mathcal{T}_{N,m} \to \mathbb{R}$, as

$$\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle = \sum_{i_1=1}^{m} \sum_{i_2=1}^{m} \ldots \sum_{i_N=1}^{m} a_{i_1,i_2,\ldots,i_N} b_{i_1,i_2,\ldots,i_N}. \tag{1}$$

Let us denote by $\| \cdot \|$ the norm endowed by the inner product (1).

Let us remark that Definition 1 extends the usual inner product for matrices. Indeed if $\mathbf{A}$ and $\mathbf{B}$ are tensors of order two, i.e. matrices, the inner product defined above coincides with the usual one on the space of matrices, that is

$$\langle\langle \mathbf{A}, \mathbf{B} \rangle\rangle = \mathrm{tr}(\mathbf{A}\mathbf{B}^T),$$

where tr(·) is the trace operator and ^{T} denotes the transpose operation. Moreover, the norm induced by the inner product for tensors is the analogous one of the Frobenius norm for matrices.

We recall some concepts related to the monotonicity of tensor functions.

Definition 2 Let $K \subset \mathcal{T}_{N,m}$. A tensor function $F : K \to \mathcal{T}_{N,m}$ is said to be

- monotone on K if, for each $\mathbf{A}, \mathbf{B} \in \mathcal{T}_{N,m}$,

$$\langle\langle F(\mathbf{A}) - F(\mathbf{B}), \mathbf{A} - \mathbf{B}\rangle\rangle \geq 0$$

- strictly monotone on K if, for each $\mathbf{A}, \mathbf{B} \in \mathcal{T}_{N,m}$ with $\mathbf{A} \neq \mathbf{B}$,

$$\langle\langle F(\mathbf{A}) - F(\mathbf{B}), \mathbf{A} - \mathbf{B}\rangle\rangle > 0$$

- strongly monotone on K if, for each $\mathbf{A}, \mathbf{B} \in \mathcal{T}_{N,m}$, there exists $k > 0$ such that

$$\langle\langle F(\mathbf{A}) - F(\mathbf{B}), \mathbf{A} - \mathbf{B}\rangle\rangle \geq k\|\mathbf{A} - \mathbf{B}\|^2$$

We focus the attention on the following tensor variational inequality problem.

Definition 3 Let $K \subset \mathcal{T}_{N,m}$ be a nonempty closed convex subset and let $F : K \to \mathcal{T}_{N,m}$ be a tensor operator. The tensor variational inequality (shortly TVI(F,K)) is the problem of finding $\mathbf{A} \in K$ such that:

$$\langle\langle F(\mathbf{A}), \mathbf{B} - \mathbf{A}\rangle\rangle \geq 0, \quad \forall \mathbf{B} \in K. \tag{2}$$

An existence theorem for solutions to (2) was proved in [10].

Theorem 1 *Let K be a nonempty compact convex subset of $\mathcal{T}_{N,m}$ and let $F : K \to \mathcal{T}_{N,m}$ be a continuous tensor function. Then tensor variational inequality problem* (2) *admits at least one solution.*

Another existence result for coercive tensor map was established also in [10].

Theorem 2 *Let K be a nonempty closed convex subset of $\mathcal{T}_{N,m}$ and let $F : K \to \mathcal{T}_{N,m}$ be a tensor continuous function satisfying the coercivity condition*

$$\lim_{\|\mathbf{A}\| \to +\infty} \frac{\langle\langle F(\mathbf{A}) - F(\mathbf{A}_0), \mathbf{A} - \mathbf{A}_0\rangle\rangle}{\|\mathbf{A} - \mathbf{A}_0\|} = +\infty,$$

for some $\mathbf{A}_0 \in K$. Then tensor variational inequality (2) *admits a solution.*

We are interested in the study of the set of solutions to Problem (2). To this aim, we show the following result under monotonicity assumption.

Theorem 3 *Let K be a nonempty closed convex subset of $\mathcal{T}_{N,m}$ and let $F : K \to \mathcal{T}_{N,m}$ be a continuous tensor function. It results:*

(a) *if F is monotone, then the set of solution $Sol(F, K)$ to* (2) *is closed and convex;*
(b) *if F is strictly monotone, then if there exists a solution to the tensor variational inequality* (2)*, then it is unique;*
(c) *if F is strongly monotone, then there exists a unique solution to the tensor variational inequality* (2).

Proof (a) The proof is omitted for its simplicity.
(b) See Theorem 4.5 of [10].

(c) See Theorem 4.6 of [10].

□

We remark that the minimal conditions in Theorems 1 and 3 are that K is closed and F continuous. In order to guarantee the existence some hypotheses must be added. While in Theorem 1 we assume that the set K is bounded (and then compact) but no more is required on F, in Theorem 3 K can be unbounded but F needs additional properties: strong monotonicity. Only monotonicity or strictly monotonicity does not guarantee the existence of the solution.

A more general concept of monotonicity can be considered. As Karamardian, we can generalize which he called pseudomonotonicity, as in the following definition.

Definition 4 Let $K \subset \mathcal{T}_{N,m}$. A tensor function $F : K \to \mathcal{T}_{N,m}$ is said to be pseudomonotone in the sense of Karamardian (K-pseudomonotone) iff for all $\mathbf{A}, \mathbf{B} \in K$,

$$\langle\langle F(\mathbf{A}), \mathbf{A} - \mathbf{B}\rangle\rangle \geq 0 \Rightarrow \langle\langle F(\mathbf{B}), \mathbf{A} - \mathbf{B}\rangle\rangle \geq 0$$

We can also consider something less than continuity, as in the following definition.

Definition 5 Let $K \subset \mathcal{T}_{N,m}$. A tensor function $F : K \to \mathcal{T}_{N,m}$ is said to be lower hemicontinuous along line segments, iff the function

$$\mathbf{A} \mapsto \langle\langle F(\mathbf{A}), \mathbf{B} - \mathbf{C}\rangle\rangle$$

is lower semicontinuous on the line segment $[\mathbf{B}, \mathbf{C}]$, for all $\mathbf{B}, \mathbf{C} \in K$.

So, in the bounded case, Theorem 1 can be refined in the following sense.

Theorem 4 *If K is convex closed bounded subset of $\mathcal{T}_{N,m}$ and F is K-pseudomonotone and lower hemicontinuous along line segments, then* (2) *admits a solution.*

Proof Since K is bounded, we can assume, without loss of generality, that there exists $\mathbf{C} \in K$ and $R > \|\mathbf{C}\|$ such that

$$\langle\langle F(\mathbf{A}), \mathbf{A} - \mathbf{C}\rangle\rangle > 0, \quad \forall \mathbf{A} \in K \cap \partial K_R, \tag{3}$$

where $K_R = \{\mathbf{A} \in K : \|\mathbf{A}\| \leq R\}$. Moreover, applying Theorem 2 and Theorem 3 in [35] in the particular case of single valued functions, since F is K-pseudomonotone and lower hemicontinuous along line segments, there exists $\mathbf{A}_R \in K_R$ such that

$$\langle\langle F(\mathbf{A}_R), \mathbf{A} - \mathbf{A}_R\rangle\rangle \geq 0, \quad \forall \mathbf{A} \in K_R. \tag{4}$$

We observe now that $\|\mathbf{A}_R\| < R$, indeed if $\|\mathbf{A}_R\| = R$ then (4) written with $\mathbf{A} = \mathbf{A}_R$ contradicts (3). We fix $\mathbf{B} \in K$ and for $t \in [0, 1]$ we consider the point

$$\mathbf{B}_t = (1 - t)\mathbf{A}_R + t\mathbf{B}$$

which, for t small enough belongs to K_R, concluding that

$$\langle\langle F(\mathbf{A}_R), \mathbf{B} - \mathbf{A}_R\rangle\rangle \geq 0, \quad \forall \mathbf{B} \in K.$$

□

We conclude this preliminaries recalling the notion of convergence for subsets of a given metric space (X, d), which was introduced in the 50's by Kuratowski (see [22], see also [31,40,41]).

Let $(\mathbb{K}_n)_{n\in\mathbb{N}}$ be a sequence of subsets of X. Recall that

$$d - \underline{\lim}_n \mathbb{K}_n = \left\{x \in X : \exists (x_n)_{n\in\mathbb{N}} \text{ eventually in } \mathbb{K}_n \text{ such that } x_n \xrightarrow{d} x\right\},$$

and

$$d - \overline{\lim}_n \mathbb{K}_n = \left\{ x \in X : \exists (x_n)_{n\in\mathbb{N}} \text{ frequently in } \mathbb{K}_n \text{ such that } x_n \stackrel{d}{\to} x \right\},$$

where *eventually* means that there exists $\delta \in \mathbb{N}$ such that $x_n \in \mathbb{K}_n$ for any $n \geq \delta$, and *frequently* means that there exists an infinite subset $N \subseteq \mathbb{N}$ such that $x_n \in \mathbb{K}_n$ for any $n \in N$ (in this last case, according to the notation given above, we also write that there exists a subsequence $(x_{k_n})_{n\in\mathbb{N}} \subseteq (x_n)_{n\in\mathbb{N}}$ such that $x_{k_n} \in \mathbb{K}_{k_n}$).

Finally we are now able to recall the Kuratowski's convergence of sets.

Definition 6 We say that $(\mathbb{K}_n)$ converges to some subset $\mathbb{K} \subseteq X$ in Kuratowski's sense, and we briefly write $\mathbb{K}_n \to \mathbb{K}$, if $d - \underline{\lim}_n \mathbb{K}_n = d - \overline{\lim}_n \mathbb{K}_n = \mathbb{K}$.

Thus, in order to verify that $\mathbb{K}_n \to \mathbb{K}$, it suffices to check that

- $d - \overline{\lim}_n \mathbb{K}_n \subseteq \mathbb{K}$, i.e. for any sequence $(x_n)_{n\in\mathbb{N}}$ frequently in $\mathbb{K}_n$ such that $x_n \stackrel{d}{\to} x$ for some $x \in S$, then $x \in \mathbb{K}$;
- $\mathbb{K} \subset d - \underline{\lim}_n \mathbb{K}_n$, i.e. for any $x \in \mathbb{K}$ there exists a sequence $(x_n)_{n\in\mathbb{N}}$ eventually in $\mathbb{K}_n$ such that $x_n \stackrel{d}{\to} x$.

3 Convergence of solutions to tensor variational inequalities

The tensor variational inequality problem $TVI(F, K)$ is called well-posed if it has a unique minimal norm solution that depends continuously on perturbation of F and K. Otherwise the problem is called ill-posed. We denote by $Sol(F, K)$ the set of all the solutions to the tensor variational inequality $TVI(F, K)$. As said in the previous section, existence of a solution to $TVI(F, K)$ has been studied in the case in which F is a coercive operator and K a bounded convex set. Results have been obtained also for coercive operators and not bounded convex sets, pseudo-monotone operators and bounded convex sets and as in the Hilbertian setting, one could imagine other situations interchanging the properties of F and K.

These problems are generally ill-posed and a common strategy to deal with them coming from the Hilbert case is the use of the so-called regularization methods. The central idea of these methods is to regularize the operator F into $TVI(F, K)$ with operators which guarantee the existence and uniqueness of solutions to the regularized problems and, then, obtain the converge of such solutions to the one to the ill-posed problem.

Our aim is to extend the regularization methods to the tensorial case. We first observe that if the problem $TVI(F, K)$ is ill-posed (nonlinear) and F is monotone, the regularization is direct by replacing the ill-posed problem with F by the well-posed problem with $F_\varepsilon = (F + \varepsilon I)$. Then the perturbed tensor variational inequality considered is

$$\langle F(\mathbf{A}_\varepsilon^\eta) + \varepsilon \mathbf{A}_\varepsilon^\eta, \mathbf{B} - \mathbf{A}_\varepsilon^\eta \rangle \geq 0, \quad \forall \mathbf{B} \in K_\eta, \tag{5}$$

where $\mathbf{A}_\varepsilon^\eta \in K_\eta$ and we prove the following:

Theorem 5 *Let $\varepsilon \to 0$, $\eta = o(\varepsilon)$ and $\lim_{\eta\to 0} K_\eta = K$. If F is a Lipschitz continuous operator and either K is bounded or F is coercive, there exists a constant $M > 0$ such that*

$$\|\mathbf{A}_\varepsilon^\eta\| \leq M, \quad \textit{for all } \varepsilon > 0, \eta > 0.$$

Moreover the sequence of solutions $\{\mathbf{A}_\varepsilon^\eta\}$ to (5) converges strongly to the solution $\mathbf{A}_0$ as $\varepsilon \to 0$, where $\mathbf{A}_0$ is the element of minimal norm of $Sol(F, K)$.

Proof We divide the proof in some steps.

Step 1 We prove that there exists a constant $M > 0$ such that $\|\mathbf{A}_\varepsilon^\eta\| \le M$ for all $\varepsilon > 0$. Since $\mathbf{A}_\varepsilon^\eta$ is the solution to (5) and $Pr_{K_\eta}\mathbf{A}_0 \in K_\eta$ then

$$\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), \mathbf{A}_\varepsilon^\eta - Pr_{K_\eta}\mathbf{A}_0\rangle\rangle \le 0.$$

Hence,

$$\begin{aligned}\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle &\le \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{A}_0 - \mathbf{A}_0\rangle\rangle \\ &\le \|F_\varepsilon(\mathbf{A}_\varepsilon^\eta)\|\|Pr_{K_\eta}\mathbf{A}_0 - \mathbf{A}_0\|. \end{aligned} \tag{6}$$

Since F is Lipschitz continuous, we have

$$\|F(\mathbf{A}_\varepsilon^\eta)\| \le \|F(\mathbf{A}_0)\| + \beta\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\|.$$

By Mosco's convergence of $\{K_\eta\}$, (6) becomes

$$\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \le (\|F(\mathbf{A}_0)\| + \beta\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\| + \varepsilon\|\mathbf{A}_\varepsilon^\eta\|)o(\varepsilon).$$

Subtracting $\langle\langle F_\varepsilon(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle$ to both sides of the inequality above, it results

$$\begin{aligned}\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta) - F_\varepsilon(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle &\le (\|F(\mathbf{A}_0)\| + \beta\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\| + \varepsilon\|\mathbf{A}_\varepsilon^\eta\|)o(\varepsilon) \\ &\quad + (\|F(\mathbf{A}_0)\| + \varepsilon\|\mathbf{A}_0\|)\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\|. \end{aligned} \tag{7}$$

It holds

$$\langle\langle F_\varepsilon\mathbf{A}_\varepsilon^\eta) - F_\varepsilon(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \ge \langle\langle F(\mathbf{A}_\varepsilon^\eta) - F(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle. \tag{8}$$

Combining (7) and (8) and dividing by $\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\|$, we obtain

$$\begin{aligned}\langle\langle F(\mathbf{A}_\varepsilon^\eta) - F(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle &\le \left(\frac{\|F(\mathbf{A}_0)\|}{\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\|} + \beta + \frac{\varepsilon\|\mathbf{A}_\varepsilon^\eta\|}{\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\|}\right)o(\varepsilon) \\ &\quad + (\|F(\mathbf{A}_0)\| + \varepsilon\|\mathbf{A}_0\|). \end{aligned} \tag{9}$$

If $\{\mathbf{A}_\varepsilon^\eta\}$ is not uniformly bounded, the right hand side of (9) is finite while the left hand side tends to $+\infty$. So $\{\mathbf{A}_\varepsilon^\eta\}$ is uniformly bounded, concluding Step 1.

Step 2 We prove that, if $\mathbf{A}_\varepsilon^\eta \rightharpoonup \mathbf{C}$, $\mathbf{C} \in K$, then $\mathbf{C} \in Sol(F, K)$. By monotonicity of F_ε, $\mathbf{A}_\varepsilon^\eta$ is the solution to (5) and since $Pr_{K_\eta}\mathbf{B} \in K_\eta$ for all $\mathbf{B} \in K$, it results

$$\begin{aligned} 0 &\le \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{B} - \mathbf{A}_\varepsilon^\eta\rangle\rangle \\ &\le \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), \mathbf{B} - \mathbf{A}_\varepsilon^\eta\rangle\rangle + \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{B} - \mathbf{B}\rangle\rangle \\ &\le \langle\langle F_\varepsilon(\mathbf{B}), \mathbf{B} - \mathbf{A}_\varepsilon^\eta\rangle\rangle + \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{B} - \mathbf{B}\rangle\rangle. \end{aligned}$$

As $\varepsilon \to 0$, we obtain

$$0 \le \langle\langle F(\mathbf{B}), \mathbf{B} - \mathbf{C}\rangle\rangle, \quad \forall \mathbf{B} \in K.$$

Taking, for $\mathbf{H} \in K$, $0 < t < 1$, $\mathbf{B} = \mathbf{C} + t(\mathbf{H} - \mathbf{C})$, we have

$$0 \le \langle\langle F(\mathbf{C} + t(\mathbf{H} - \mathbf{C})), \mathbf{H} - \mathbf{C}\rangle\rangle.$$

Letting $t \to 0$, $\mathbf{C} \in Sol(F, K)$, since

$$\langle\langle F(\mathbf{C}), \mathbf{H} - \mathbf{C}\rangle\rangle \ge 0, \quad \forall \mathbf{H} \in K,$$

concluding Step 2.

Step 3 We prove that if $\mathbf{A}_\varepsilon^\eta \rightharpoonup \mathbf{A}_0$, then $\mathbf{A}_\varepsilon^\eta \to \mathbf{A}_0$, as $\varepsilon \to 0$, i.e. $\|\mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\| \to 0$. Since $\langle\langle \mathbf{A}_0, \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \to 0$, it remains to prove that

$$\langle\langle \mathbf{A}_\varepsilon^\eta, \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \to 0.$$

Since $Pr_K \mathbf{A}_\varepsilon^\eta \in K$, it follows

$$\langle\langle F(\mathbf{A}_0), Pr_K \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \geq 0$$

which implies

$$\langle\langle F(\mathbf{A}_0), \mathbf{A}_0 - \mathbf{A}_\varepsilon^\eta\rangle\rangle \leq \langle\langle F(\mathbf{A}_0), Pr_K \mathbf{A}_\varepsilon^\eta - \mathbf{A}_\varepsilon^\eta\rangle\rangle. \tag{10}$$

Similarly, since $Pr_{K_\eta}\mathbf{A}_0 \in K_\eta$, it results

$$\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{A}_0 - \mathbf{A}_\varepsilon^\eta\rangle\rangle \geq 0,$$

Hence, it follows

$$\langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \leq \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{A}_0 - \mathbf{A}_0\rangle\rangle. \tag{11}$$

Adding (10) and (11), it holds

$$\begin{aligned}
&\langle\langle F(\mathbf{A}_\varepsilon^\eta) - F(\mathbf{A}_0), \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle + \varepsilon\langle\langle \mathbf{A}_\varepsilon^\eta, \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle \\
&\quad \leq \langle\langle F(\mathbf{A}_0), Pr_K \mathbf{A}_\varepsilon^\eta - \mathbf{A}_\varepsilon^\eta\rangle\rangle + \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{A}_0 - \mathbf{A}_0\rangle\rangle \\
&\quad \leq \langle\langle F(\mathbf{A}_0), Pr_K \mathbf{A}_\varepsilon^\eta - Pr_{K-\eta}\mathbf{A}_\varepsilon^\eta\rangle\rangle + \langle\langle F_\varepsilon(\mathbf{A}_\varepsilon^\eta), Pr_{K_\eta}\mathbf{A}_0 - Pr_K\mathbf{A}_0\rangle\rangle \\
&\quad \leq \|F(\mathbf{A}_0)\|\|Pr_K \mathbf{A}_\varepsilon^\eta - Pr_{K_\eta}\mathbf{A}_\varepsilon^\eta\| + \|F_\varepsilon(\mathbf{A}_\varepsilon^\eta)\|\|Pr_{K_\eta}\mathbf{A}_0 - Pr_K\mathbf{A}_0\| \\
&\quad = o(\varepsilon).
\end{aligned}$$

Then, we have

$$\lim_{\varepsilon \to 0} \langle\langle \mathbf{A}_\varepsilon^\eta, \mathbf{A}_\varepsilon^\eta - \mathbf{A}_0\rangle\rangle = 0,$$

concluding Step 3.

It remains to prove that $\mathbf{A}_0$ is the element of minimum norm in $Sol(F, K)$. By Step 2 there exists a sequence $\mathbf{A}_\nu \rightharpoonup \mathbf{C} \in Sol(F, K)$. For any $\mathbf{H} \in Sol(F, K)$, since $Pr_{K_\eta}\mathbf{H} \in K_\eta$ and $Pr_K\mathbf{H} \in K$, we have

$$\begin{aligned}
&\langle\langle F_\nu(\mathbf{A}_\nu), Pr_{K_\eta}\mathbf{H} - \mathbf{A}_\nu\rangle\rangle \geq 0, \\
&\langle\langle F(\mathbf{H}), \mathbf{A}_\nu - \mathbf{H}\rangle\rangle \geq \langle\langle F(\mathbf{H}), \mathbf{A}_\nu - Pr_K\mathbf{H}\rangle\rangle,
\end{aligned}$$

and

$$\langle\langle F_\nu(\mathbf{A}_\nu), \mathbf{A}_\nu - \mathbf{H}\rangle\rangle \leq \langle\langle F_\nu(\mathbf{A}_\nu), Pr_{K_\eta}\mathbf{H} - \mathbf{H}\rangle\rangle.$$

Adding last two inequalities, as in Step 3, we have

$$\langle\langle F_\nu(\mathbf{A}_\nu) - F(\mathbf{H}), \mathbf{A}_\nu - \mathbf{H}\rangle\rangle \leq \|F(\mathbf{H})\|\|Pr_K\mathbf{A}_\nu - \mathbf{A}_\nu\| + \|F_\nu(\mathbf{A}_\nu)\|\|Pr_{K_\eta}\mathbf{H} - \mathbf{H}\|.$$

Since $\langle\langle F(\mathbf{A}_\nu) - F(\mathbf{H}), \mathbf{A}_\nu - \mathbf{H}\rangle\rangle \geq 0$,

$$\nu\langle\langle \mathbf{A}_\nu, \mathbf{A}_\nu - \mathbf{H}\rangle\rangle \leq \|F(\mathbf{H})\|\|Pr_K\mathbf{A}_\nu - \mathbf{A}_\nu\| + \|F_\nu(\mathbf{A}_\nu)\|\|Pr_{K_\eta}\mathbf{H} - \mathbf{H}\|,$$

which implies, as $\nu \to 0$,

$$\langle\langle \mathbf{C}, \mathbf{C} - \mathbf{H}\rangle\rangle \leq 0, \quad \forall\, \mathbf{H} \in Sol(F, K).$$

Having

$$\|\mathbf{C}\|^2 \leq \langle\langle \mathbf{C}, \mathbf{H} \rangle\rangle \leq \|\mathbf{C}\| \|\mathbf{H}\|, \quad \forall\ \mathbf{H} \in Sol(F, K),$$

and by strong convergence proved in Step 3, we obtain that $\mathbf{C}$ is the element of minimum norm. □

4 Stability results

In this section we study the sensitivity analysis about solutions to a tensor variational inequality $TVI(, K)$. In particular, in the following theorems we study the stability of solutions under small perturbations of the tensor function F.

We present now a theorem about the sensitivity of solutions. The following result establishes that in the strong monotone case, a small change of the function produces a small change in the solutions.

Theorem 6 *Let $F : K \to \mathcal{T}_{N,m}$ be a strongly monotone tensor function with constant α and let $\tilde{F} : K \to \mathcal{T}_{N,m}$ a perturbation of F. If we denote by $\mathbf{T}^*$ and $\tilde{\mathbf{T}}$ the correspondent solutions of the following tensor variational inequalities:*

$$\langle\langle F(\mathbf{T}^*), \mathbf{B} - \mathbf{T}^* \rangle\rangle \geq 0, \quad \forall \mathbf{B} \in K. \tag{12}$$

$$\langle\langle \tilde{F}(\tilde{\mathbf{T}}), \mathbf{B} - \tilde{\mathbf{T}} \rangle\rangle \geq 0, \quad \forall \mathbf{B} \in K. \tag{13}$$

Then it follows

$$\|\mathbf{T}^* - \tilde{\mathbf{T}}\| \leq \frac{1}{\alpha} \|\tilde{F}(\tilde{\mathbf{T}}) - F(\tilde{\mathbf{T}})\|.$$

Proof If we choose $\mathbf{B} = \tilde{\mathbf{T}}$ in (12) and $\mathbf{B} = \mathbf{T}^*$ in (13), by summing up the two inequalities, we obtain:

$$\langle\langle \tilde{F}(\tilde{\mathbf{T}}) - F(\mathbf{T}^*), \mathbf{T}^* - \tilde{\mathbf{T}} \rangle\rangle \geq 0.$$

If now we add and subtract $F(\tilde{\mathbf{T}})$ in the previous inequality, we obtain

$$\langle\langle \tilde{F}(\tilde{\mathbf{T}}) - F(\tilde{\mathbf{T}}), \mathbf{T}^* - \tilde{\mathbf{T}} \rangle\rangle \geq \langle\langle F(\mathbf{T}^*) - F(\tilde{\mathbf{T}}), \mathbf{T}^* - \tilde{\mathbf{T}} \rangle\rangle.$$

Recalling the strong monotonicity of F and the Cauchy–Schwartz inequality, we obtain

$$\begin{aligned} \alpha\|\mathbf{T}^* - \tilde{\mathbf{T}}\|^2 &\leq \langle\langle F(\mathbf{T}^*) - F(\tilde{\mathbf{T}}), \mathbf{T}^* - \tilde{\mathbf{T}} \rangle\rangle \\ &\leq \langle\langle \tilde{F}(\tilde{\mathbf{T}}) - F(\tilde{\mathbf{T}}), \mathbf{T}^* - \tilde{\mathbf{T}} \rangle\rangle \\ &\leq \|\tilde{F}(\tilde{\mathbf{T}}) - F(\tilde{\mathbf{T}})\| \cdot \|\mathbf{T}^* - \tilde{\mathbf{T}}\|, \end{aligned}$$

concluding that

$$\|\mathbf{T}^* - \tilde{\mathbf{T}}\| \leq \frac{1}{\alpha} \|\tilde{F}(\tilde{\mathbf{T}}) - F(\tilde{\mathbf{T}})\|.$$

□

A more refined analysis with semi-coercive functions can be obtained applying Theorem 4.1 in [1] to tensors. Firstly we present some definitions.

Definition 7 An operator $F : \mathcal{T}_{N,m} \to \mathcal{T}_{N,m}$ is called semi-coercive if it satisfies

$$\langle\langle F(\mathbf{A}) - F(\mathbf{B}), \mathbf{A} - \mathbf{B}\rangle\rangle \geq k(\text{dist}_U(\mathbf{A} - \mathbf{B}))^2$$
$$F(\mathbf{T} + \mathbf{B}) = F(\mathbf{T}), \quad \forall \mathbf{T} \in \mathcal{T}_{N,m}, \ \mathbf{B} \in U, \text{and } F(\mathcal{T}_{N,m}) \subset \overset{\circ}{U},$$

for some $k > 0$ and some closed subspace $U \subset H$.

We observe that the projection operator onto a closed subspace of a Hilbert space is an example of semi-coercive function.

Definition 8 A set $K \subset \mathcal{T}_{N,m}$ is well-positioned if there exist $\mathbf{T}_0 \in \mathcal{T}_{N,m}$ and $\mathbf{T} \in \mathcal{T}_{N,m}$ such that

$$\langle\langle \mathbf{T}, \mathbf{T} - \mathbf{T}_0\rangle\rangle \geq \|\mathbf{T} - \mathbf{T}_0\|, \quad \forall \mathbf{T} \in K.$$

Since a set is well-positioned if and only if its convex hull is well-positioned, we can consider, with loss of generality, only convex sets.

Definition 9 The epigraph of a tensor function $F : \mathcal{T}_{N,m} \to \mathbb{R}$ is the set of points on or above its graph:

$$epi F = \{(\mathbf{T}, y) \ : \ \mathbf{T} \in \mathcal{T}_{N,m}, \ y \geq F(\mathbf{T})\}$$

Definition 10 The recession cone of a closed convex set K is the maximal convex cone whose translate in every point of K lies in K:

$$K_\infty = \{\mathbf{T} \in K \ : \ \forall \alpha > 0, \ \mathbf{T}_0 \in K, \ \mathbf{T}_0 + \alpha \mathbf{T} \in K\}.$$

The recession function of a proper lower semi-continuous function f is the proper lower semi-continuous function f_∞ whose epigraph is the recession cone for the epigraph of f, i.e. $epi f_\infty = (epi f)_\infty$.

Let us define the following function

$$\Psi(\mathbf{T}) = k(\text{dist}_U(\mathbf{T}))^2 + I_K(\mathbf{T}), \qquad \forall \mathbf{T} \in \mathcal{T}_{N,m},$$

where I_K is the indicator function of the set K.

We can apply Theorem 4.1 in [1] in a suitable way obtaining the following theorem for semi-coercive tensor functions.

Theorem 7 *Let $F : \mathcal{T}_{N,m} \to \mathcal{T}_{N,m}$ be a bounded semi-coercive operator on $\mathcal{T}_{N,m}$ and K be a nonempty closed convex set. The following two statement are equivalent*

1. *there is $\varepsilon > 0$ such that $Sol(F_\varepsilon, K_\varepsilon) \neq \emptyset$ if*

$$\|F(\mathbf{T}) - F_\varepsilon(\mathbf{T})\|_{\mathbf{T}^*} < \varepsilon, \ \ \forall \mathbf{T} \in H,$$
$$K \subset K_\varepsilon + \varepsilon B \ \textit{and} \ K_\varepsilon \subset K + \varepsilon B;$$

2. *the following two conditions hold*
   - *the set $epi\Psi$ is well-positioned;*
   - $\Psi_\infty(\mathbf{T}) > 0, \quad \forall \mathbf{T} \in K_\infty, \mathbf{T} \neq 0.$

5 General oligopolistic market equilibrium problem

In [10], a general oligopolistic market equilibrium model in which every firm produces several goods is presented.

Let us describe the model. Let us consider:

- m firms P_i, $i = 1, \dots, m$;
- n demand markets Q_j, $j = 1, \dots, n$;

and assume that every firm P_i produces a certain number l of different commodities.

Firm P_i and demand market Q_j are generally spatially separated, for all $i = 1, \dots, m$, $j = 1, \dots, n$. Let x_{ij}^k, $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, l$, be the nonnegative variable expressing the commodity shipment of kind k between the producer P_i and the market Q_j. Observe that the nonnegative commodity shipment for all the goods between the producers and the demand markets belongs to $\mathcal{T}(\mathbb{R}_+)$. Furthermore, we assume that the nonnegative commodity shipment x_{ij}^k has to satisfy the following constraints

$$\underline{x}_{ij}^k \leq x_{ij}^k \leq \overline{x}_{ij}^k, \quad \forall i = 1, \dots, m, \ \forall j = 1, \dots, n, \ \forall k = 1, \dots, l,$$

where $\underline{x}_{ij}^k, \overline{x}_{ij}^k$ are nonnegative bounds belonging to $\mathcal{T}(\mathbb{R}_+)$.

Let us consider also the following variables:

- $p_i^k, i = 1, \dots, m, k = 1, \dots, l$ which expresses the commodity output of kind k produced by the firm P_i;
- q_j^k, $j = 1, \dots, n$, $k = 1, \dots, l$ which expresses the demand for the commodity of kind k of demand market Q_j

Assume that both variables p_i^k and q_j^k are nonnegative.

Let us suppose that the following feasible conditions hold:

$$p_i^k = \sum_{j=1}^{n} x_{ij}^k, \quad i = 1, \dots, m, \ k = 1, \dots, l, \tag{14}$$

$$q_j^k = \sum_{i=1}^{m} x_{ij}^k, \quad j = 1, \dots, n, \ k = 1, \dots, l. \tag{15}$$

We can read the above conditions in this way: the quantity produced by each firm P_i of kind k must be equal to the commodity shipments of such kind from that firm to all the demand markets. Also the quantity demanded by each demand market Q_j of kind k must be equal to the commodity shipments of such kind from all the firms to that demand market. As a consequence, the total production p_i by the firm P_i and the total demand q_j of the demand market Q_j are given by

$$p_i = \sum_{k=1}^{l} \sum_{j=1}^{n} x_{ij}^k, \quad i = 1, \dots, m,$$

$$q_j = \sum_{k=1}^{l} \sum_{i=1}^{m} x_{ij}^k, \quad j = 1, \dots, n,$$

respectively.

Hence, the feasible set is given by

$$\mathbb{K} = \Big\{ x \in \mathcal{T}(\mathbb{R}_+) : \\ \underline{x}_{ij}^k \le x_{ij}^k \le \overline{x}_{ij}^k, \quad \forall i = 1, \ldots, m,\ \forall j = 1, \ldots, n,\ \forall k = 1, \ldots, l \Big\} \tag{16}$$

where $\underline{x}$, $\overline{x}$ are nonnegative bounds belonging to $\mathcal{T}(\mathbb{R}_+)$. Let us note that $\mathbb{K}$ is a convex, closed and bounded set of the Hilbert space $\mathcal{T}(\mathbb{R}_+)$.

In our model we also consider the following functions depending on the commodity shipments:

- $f_i^k : \mathcal{T} \to \mathcal{T}$, $i = 1, \ldots, m$, $k = 1, \ldots, l$, which denotes the production cost of P_i for each good of type k;
- $d_j^k : \mathcal{T} \to \mathcal{T}$, $j = 1, \ldots, n$, $k = 1, \ldots, l$, which denotes the demand price for unity of kind k of the demand market Q_j;
- $c_{ij}^k : \mathcal{T} \to \mathcal{T}$, $i = 1, \ldots, m$, $j = 1, \ldots, n,$, $k = 1, \ldots, l$, which denotes the tensor variable expressing the transaction cost between firm P_i and demand market Q_j regarding the good of kind k.

Hence, the profit v_i of the firm P_i, $i = 1, \ldots, m$, is given by

$$v_i(x) = \sum_{k=1}^{l} \left[\sum_{j=1}^{n} d_j^k(x) x_{ij}^k - f_i^k(x) - \sum_{j=1}^{n} c_{ij}^k(x) x_{ij}^k \right],$$

i.e. the difference between the price that each demand market P_i is disposed to pay and the sum of the production costs minus the transportation costs.

In our model the m firms, each of them producing different kind of goods, supply the commodity in a noncooperative fashion, each one trying to maximize its own profit function considered the optimal distribution pattern for the other firms. The aim is to determine a nonnegative tensor commodity distribution x for which the m firms and the n demand markets will be in a state of equilibrium as defined below like generalizing the Cournot–Nash equilibrium principle.

Definition 11 A feasible tensor function $x^* \in \mathbb{K}$ is a general oligopolistic market equilibrium distribution if and only if, for each $i = 1, \ldots, m$, it results

$$v_i(x^*) \ge v_i(x_i, \hat{x}_i^*), \tag{17}$$

where $\hat{x}_i^* = (x_1^*, \ldots, x_{i-1}^*, x_{i+1}^*, \ldots, x_m^*)$ and x_i is a slice of dimension nl.

Let us consider the function

$$\nabla_D v = \left(\frac{\partial v_i}{\partial x_{ij}^k} \right)_{ijk}, \qquad i = 1, \ldots, m,\ \ j = 1, \ldots, n,\ \ k = 1, \ldots, l,$$

that is the tensor of partial derivatives of v with respect to the tensor variables x_{ij}^k.

In order to derive an equivalent formulation of Definition 11 with a suitable tensor variational inequality, let us suppose the following assumptions:

i. $v(x)$ is continuously differentiable with respect to x,

ii. $v_i(x)$ is pseudoconcave with respect to the variables x_i, namely the following condition holds (see [6])

$$\left\langle\left\langle \frac{\partial v_i}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_m), x_i - y_i \right\rangle\right\rangle \geq 0$$
$$\Rightarrow v_i(x_1, \ldots, x_i, \ldots, x_m) \geq v_i(x_1, \ldots, y_i, \ldots, x_m),$$

Under assumptions (i) and (ii) on v_i, we establish the following variational formulation (see [10]).

Theorem 8 *Let us suppose that assumptions (i) and (ii) are satisfied. Then, $x^* \in \mathbb{K}$ is a general oligopolistic market equilibrium distribution according to Definition 11 if and only if it satisfies the tensor variational inequality*

$$\langle\langle -\nabla_D v(x^*), x - x^* \rangle\rangle = -\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l} \frac{\partial v_i(x^*)}{\partial x_{ij}^k}(x_{ij}^k - (x_{ij}^k)^*) \geq 0, \quad \forall x \in \mathbb{K}. \tag{18}$$

We can now derive theorems about existence and stability of solutions to the general oligopolistic market equilibrium problem. Firstly, we can obtain the following theorem adapting existence theorems for tensor variational inequalities and taking into account that our constraint set K is nonempty bounded closed and convex.

Theorem 9 *Let us suppose that assumptions (i) and (ii) are satisfied. Moreover, if $-\nabla_D v$ is strongly monotone then there exists a unique general oligopolistic market equilibrium distribution.*

The following second result follows by Theorem 6.

Theorem 10 *Assume that the profit function changes from $v(\cdot)$ to the perturbed function $\tilde{v}(\cdot)$ and denote by x^* and $\tilde{x}$ the correspondent solutions to the associated tensor variational inequalities:*

$$\langle\langle -\nabla_D v(x^*), x - x^* \rangle\rangle \geq 0, \quad \forall x \in \mathbb{K},$$
$$\langle\langle -\nabla_D \tilde{v}(\tilde{x}), x - \tilde{x} \rangle\rangle \geq 0, \quad \forall x \in \mathbb{K}.$$

If $-\nabla_D v$ is a strongly monotone function of constant α, then

$$\|x^* - \tilde{x}\| \leq \frac{1}{\alpha}\| - \nabla_D \tilde{v}(\tilde{x}) + \nabla_D v(\tilde{x})\|$$

6 Numerical example

Let us describe here a numerical example for the general oligopolistic market equilibrium problem.

Let us consider a market network constituted by two firms P_1 and P_2 which compete with three markets Q_1, Q_2 and Q_3. We suppose that every firm P_i, $i = 1, 2$, produce two different kind of commodities. Let x_{ij}^k be the kth commodity shipment from P_i to Q_j, $(i = 1, 2,\ j = 1, \ldots, 3,\ k = 1, 2)$ and assume that the constraints $0 \leq x_{ij}^k \leq 100$ hold, for every $i = 1, 2$, $j = 1, \ldots, 3$, $k = 1, 2$.

Let p be the matrix of the commodity production:

$$p = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix},$$

and let q be the matrix of the commodity demand:

$$q = \begin{pmatrix} 2 & 3/2 \\ 0 & 3 \\ 5 & 0 \end{pmatrix}.$$

As a consequence, the feasible set is

$$\mathbb{K} = \Big\{ x \in \mathcal{T}(\mathbb{R}_+) : \\ 0 \le x_{ij}^k \le 100, \quad \forall i = 1, 2, \ \forall j = 1, 2, 3, \ \forall k = 1, 2 \Big\}.$$

Let us consider the profit functions v_i defined by

$$\begin{aligned} v_1 &= -4(x_{11}^1)^2 + 6x_{11}^1 + 2x_{11}^1 x_{13}^1 - 4x_{13}^1 \\ v_2 &= -2(x_{21}^1)^2 + 16x_{21}^1 - 2x_{21}^1 x_{11}^1 + 3x_{11}^1 + 3x_{22}^2 - 2x_{22}^2 x_{21}^1 \end{aligned}$$

and a small perturbation v_i^ε of v_i considering

$$\begin{aligned} v_1^\varepsilon &= -4(x_{11}^1)^2 + 6x_{11}^1 + \varepsilon x_{11}^1 + 2x_{11}^1 x_{13}^1 - 4x_{13}^1 \\ v_2^\varepsilon &= -2(x_{21}^1)^2 + 16x_{21}^1 - 2x_{21}^1 x_{11}^1 + 3x_{11}^1 + \varepsilon x_{11}^1 + 3x_{22}^2 - 2x_{22}^2 x_{21}^1 - 2\varepsilon x_{21}^2 \end{aligned}$$

Then, the components of ∇v different from zero of v_i^ε and v_i are given by

$$\begin{aligned} \frac{\partial v_1}{x_{11}^1} &= -8x_{11}^1 + 6 + 2x_{13}^1 & \frac{\partial v_1^\varepsilon}{x_{11}^1} &= -8x_{11}^1 + 6 + \varepsilon + 2x_{13}^1 \\ \frac{\partial v_1}{x_{13}^1} &= 2x_{11}^1 - 4 & \frac{\partial v_1^\varepsilon}{x_{13}^1} &= 2x_{11}^1 - 4 \\ \frac{\partial v_2}{x_{21}^1} &= -4x_{21}^1 + 16 - 2x_{11}^1 - 2x_{22} & \frac{\partial v_2^\varepsilon}{x_{21}^1} &= -4x_{21}^1 + 16 - 2x_{11}^1 - 2x_{22} - 2\varepsilon \\ \frac{\partial v_2}{x_{11}^1} &= -2x_{21}^1 + 3 & \frac{\partial v_2^\varepsilon}{x_{11}^1} &= -2x_{21}^1 + \varepsilon + 3 \\ \frac{\partial v_2}{x_{22}^2} &= 3 - 2x_{21}^1 & \frac{\partial v_2}{x_{22}^2} &= 3 - 2x_{21}^1 \end{aligned}$$

In order to compute the solution we make use of the direct method proposed in [28]. More precisely, we consider the following systems:

$$\begin{cases} -8x_{11}^1 + 6 + 2x_{13}^1 = 0 \\ 2x_{11}^1 - 4 = 0 \\ -4x_{21}^1 + 16 - 2x_{11}^1 - 2x_{22} = 0 \\ -2x_{21}^1 + 3 = 0 \\ 3 - 2x_{21}^1 = 0 \end{cases} \quad \begin{cases} -8x_{11}^1 + 6 + \varepsilon + 2x_{13}^1 = 0 \\ 2x_{11}^1 - 4 = 0 \\ -4x_{21}^1 + 16 - 2x_{11}^1 - 2x_{22} - 2\varepsilon = 0 \\ -2x_{21}^1 + 3 + \varepsilon = 0 \\ 3 - 2x_{21}^1 = 0 \end{cases}$$

and we obtain the solutions x_* and x_*^ε, having the following components different from zero:

$$x^1_{11} = 2 \qquad x^1_{11} = 2$$
$$x^1_{13} = 5 \qquad x^1_{13} = (10 - \varepsilon)/2$$
$$x^1_{21} = 3/2 \qquad x^1_{21} = (3 + \varepsilon)/2$$
$$x^2_{22} = 3 \qquad x^2_{22} = 3.$$

It is easy to prove that x_* and x_*^ε verify the feasible conditions, then they are general oligopolistic market equilibrium distributions for the correspondent problems. Moreover, we can compute

$$\|x_* - x_*^\varepsilon\| = \frac{1}{2}\varepsilon^2$$

and

$$\| - \nabla_D v^\varepsilon(x_*^\varepsilon) + \nabla_D v(x_*^\varepsilon)\| = 6\varepsilon^2,$$

from which it possible to verify the conclusion of Theorem 10 since the constant of strongly monotonocity for $\nabla_D v$ is greater than 1.

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